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Fidelity for displaced squeezed thermal states and the oscillator semigroup

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Abstract. The fidelity for two displaced squeezed thermal states is computed using the fact that the corresponding density operators belong to the oscillator semigroup. A novel calculation technique is developed for the computation of the traces of product of Gaussian density operators and of square roots of Gaussian density operators. The method is exemplified for the one-mode case but it is also applicable to the multimode case.

1. Introduction

The concept of fidelity is a basic ingredient in quantum communication theory [1, 2]. Recently the corresponding Bures distance was used [3] to define a measure of the entanglement as the minimal Bures distance of an entangled state to the set of disentangled states.

Let ρ_1 and ρ_2 be the density operators which describe two impure states. The natural candidate for the fidelity, denoted by $F(\rho_1, \rho_2)$, is the transition probability which must satisfy the following natural axioms:

- (F1) $F(\rho_1, \rho_2) \leq 1$ and $F(\rho_1, \rho_2) = 1$ if and only if $\rho_1 = \rho_2$;
- (F2) $F(\rho_1, \rho_2) = F(\rho_2, \rho_1)$;
- (F3) If ρ_1 is a pure state $\rho_1 = |\psi_1\rangle\langle\psi_1|$ then $F(\rho_1, \rho_2) = \langle\psi_1|\rho_2|\psi_1\rangle$;
- (F4) $F(\rho_1, \rho_2)$ is invariant under unitary transformations on the state space.

Uhlmann's transition probability for mixed states [4]

$$F(\rho_1, \rho_2) = [\text{trace}(\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^{\frac{1}{2}}]^2 \quad (1)$$

does satisfy (F1)–(F4).

Investigations into the detailed structure of fidelity has been hampered by the complicated square-root factors in (1). Due to these technical difficulties in computing fidelity, few concrete results concerning the details of the fidelity have been found. Until recently all known concrete results have been calculated only for finite dimensional Hilbert spaces [5–7]. The first result in infinite-dimensional Hilbert space has been obtained by Twamley [8] for the fidelity of two undisplaced thermal states. Twamley combines the Schur factorization with Baker–Campbell–Hausdorff identities. However, as he said these arguments do not seem to hold for displaced squeezed thermal states. We have obtained

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the fidelity for two displaced thermal states [9] using a result of Wilcox [10]. In the present paper the fidelity for two displaced squeezed thermal states is computed using the fact that the corresponding density operators belong to the oscillator semigroup.

2. The oscillator semigroup

In the following, we shall use a completely new method for the treatment of the complicated square-root factors in (1) in the case of two displaced squeezed thermal states (i.e. in the case of two displaced mixed quasi-free states [11]). The basic ingredient of the method is the oscillator semigroup [12, 13]. The oscillator semigroup is the semigroup of integral operators on $L^2(\mathbb{R})$ whose integral kernels are Gaussians (where \mathbb{R} is standard notation for the real line). The density operators which describe the displaced squeezed thermal states belong evidently to this semigroup. Indeed the oscillator semigroup (or its closure) contains the semigroup generated by the Hermite operator [13, 14] which is the Hamiltonian of the quantum oscillator (and respectively the range of metaplectic representation which generate the squeezing [15, 16]). The metaplectic representation [12, 13] is the projective (double-valued) unitary representation of the symplectic group $\text{Sp}(1, \mathbb{R})$ on $L^2(\mathbb{R})$ defined by the uniqueness of the canonical commutation relations.

The most general Gaussian density operator in coordinate representation is an integral operator

$$(\rho\psi)(x) = \int_{-\infty}^{+\infty} \langle x|\rho|y\rangle\psi(y) dy \quad (2)$$

where

$$\langle x|\rho|y\rangle = \exp[-(ax^2 + dy^2 + 2bxy) + lx + ky + g]. \quad (3)$$

In order that ρ is a quantum density operator it must be Hermitian, normalizable and non-negative [16]. Hermiticity for ρ requires $d = \bar{a}$, $b = \bar{b}$, $k = \bar{l}$ and $g = \bar{g}$ [16]. From $\text{trace } \rho = 1$ it follows that

$$g = -\frac{(\text{Re } l)^2}{2(\text{Re } a + b)} - \ln \sqrt{\frac{\pi}{2(\text{Re } a + b)}}$$

and $\text{Re } a \geq -b$. From the non-negativity of $\langle \psi|\rho|\psi\rangle$ for all $|\psi\rangle$ it follows [16] that $-b \geq 0$. Hence $\text{Re } a \geq -b \geq 0$. For two quantum density operators ρ_1 and ρ_2 we have the semigroup composition law

$$\langle x|\rho_1\rho_2|y\rangle = \int_{-\infty}^{+\infty} \langle x|\rho_1|z\rangle\langle z|\rho_2|y\rangle dz. \quad (4)$$

Let us denote by A , B , D , L , K and G the corresponding parameters of the Gaussian $\langle x|\rho_1\rho_2|y\rangle$. Then the semigroup composition law is given by the following rule

$$\text{R1} = \begin{cases} A = a_1 - \frac{b_1^2}{d_1 + a_2} & D = d_2 - \frac{b_2^2}{d_1 + a_2} \\ B = -\frac{b_1 b_2}{d_1 + a_2} & L = l_1 - \frac{(k_1 + l_2)b_1}{d_1 + a_2} \\ K = k_2 - \frac{(k_1 + l_2)b_2}{d_1 + a_2} & G = g_1 + g_2 + \frac{(k_1 + l_2)}{4(d_1 + a_2)} + \ln \sqrt{\frac{\pi}{d_1 + a_2}}. \end{cases} \quad (5)$$

We shall define the operator $\sqrt{\bar{\rho}}$ as the integral operator with the Gaussian kernel

$$\langle x|\sqrt{\bar{\rho}}|y\rangle = \exp[-(\tilde{a}x^2 + \tilde{d}y^2 + 2\tilde{b}xy) + \tilde{l}x + \tilde{k}y + \tilde{g}] \quad (6)$$

such that

$$\langle x|\rho|y\rangle = \int_{-\infty}^{+\infty} \langle x|\sqrt{\rho}|z\rangle \langle z|\sqrt{\rho}|y\rangle dz. \tag{7}$$

Then using the above given semigroup composition rule R1 one obtains by inversion the following rule for the square root of a density operator

$$R2 = \begin{cases} \tilde{a} = a - b & \tilde{d} = d - b \\ \tilde{b} = -\sqrt{-b(a + d - 2b)} \\ \text{where the minus sign was chosen in order to have } -\tilde{b} \geq 0. \\ \tilde{l} + \tilde{k} = \frac{l + k}{1 + 2\sqrt{\frac{-b}{a + d - 2b}}} & \tilde{l} - \tilde{k} = l - k \\ \tilde{g} = \frac{1}{2}g - \frac{1}{2}\ln\sqrt{\frac{\pi}{a + d - 2b}} - \frac{(l + k)^2}{8(\sqrt{a + d - 2b} + 2\sqrt{-b})^2}. \end{cases} \tag{8}$$

3. Transition probability

The characteristic function of the density operator ρ is given by [11, 16]

$$CF_\rho(\alpha, \tau) = \text{trace}(W(\alpha, \tau)\rho) = \exp[-\frac{1}{4}(a_{pp}\alpha^2 + a_{qq}\tau^2 + 2a_{pq}\alpha\tau)] \tag{9}$$

where

$$\begin{aligned} (W(\alpha, \tau)\psi)(x) &= \exp\left[i\tau\left(x - \frac{\alpha}{2}\right)\right]\psi(x - \alpha) \\ a_{qq} &= 2(\langle Q^2\rangle - \langle Q\rangle^2) = \frac{1}{a + d + 2b} \\ a_{pp} &= 2(\langle P^2\rangle - \langle P\rangle^2) = \frac{4(ad - b^2)}{a + d + 2b} \\ a_{pq} &= 2(\frac{1}{2}(\langle QP + PQ\rangle - \langle Q\rangle\langle P\rangle)) = \frac{i(a - d)}{a + d + 2b}. \end{aligned} \tag{10}$$

Here $(Q\psi)(x) = x\psi(x)$ and $(P\psi)(x) = -i(d\psi(x))/dx$.

If \mathbf{A} is the matrix $\begin{pmatrix} a_{qq} & a_{pq} \\ a_{pq} & a_{pp} \end{pmatrix}$ and $\det \mathbf{A} = a_{qq}a_{pp} - a_{pq}^2$ it is easy to show that

$$a = \frac{\det \mathbf{A} + 1}{4a_{qq}} - \frac{ia_{pq}}{2a_{qq}} \tag{11}$$

$d = \bar{a}$ and

$$b = -\frac{\det \mathbf{A} - 1}{4a_{qq}}. \tag{12}$$

In order to simplify the calculations we shall use the property (F4). Then it suffices to consider that ρ_1 is a thermal state (i.e. an undisplaced and unsqueezed state) and that only ρ_2 is a displaced squeezed thermal state. It is well known that in this case [14]

$$\langle x|\rho_1|y\rangle = \exp\left[-\frac{1}{2}\coth\beta(x^2 + y^2) + \frac{xy}{\sinh\beta} - \ln\sqrt{\frac{\pi}{\tanh\beta}}\right]. \tag{13}$$

If ρ'_2 is an undisplaced squeezed thermal state with

$$\langle x|\rho'_2|y\rangle = \exp[-(ax^2 + dy^2 + 2bxy) + g'] \tag{14}$$

then the displaced squeezed thermal state ρ_2 is obtained as $\rho_2 = W(\alpha, \tau)\rho'_2W(-\alpha, -\tau)$ and the corresponding kernel is given by $\langle x|\rho_2|y\rangle = \exp[i\tau(x - y)]\langle x - \alpha|\rho'_2|x - \alpha\rangle$, i.e.

$$\langle x|\rho_2|y\rangle = \exp[-(ax^2 + dy^2 + 2bxy) + lx + ky + g] \tag{15}$$

where $l = 2(a + d)\alpha + i\tau$, $k = \bar{l}$ and $g = g' - (a + d + 2b)\alpha^2$. Now we can use the rules R1 and R2. After long but simple calculations we obtain the main result of the paper

$$F(\rho_1, \rho_2) = \frac{2}{\sqrt{\Delta + T} - \sqrt{T}} \exp[-\mathbf{u}^T(\mathbf{A}_1 + \mathbf{A}_2)^{-1}\mathbf{u}] \tag{16}$$

where $\Delta = \det(\mathbf{A}_1 + \mathbf{A}_2)$, $T = (\det \mathbf{A}_1 - 1)(\det \mathbf{A}_2 - 1)$ and where \mathbf{u} is the column vector $\begin{pmatrix} \alpha \\ \tau \end{pmatrix}$. A Gaussian density matrix describes a pure state if and only if $\det \mathbf{A} = 1$ [11]. If ρ_1 is a pure state then [11]

$$F(\rho_1, \rho_2) = \text{trace } \rho_1\rho_2 = (2\pi)^{-1} \int_{-\infty}^{+\infty} CF_{\rho_1}(-\alpha, -\tau)CF_{\rho_2}(\alpha, \tau) d\alpha d\tau \tag{17}$$

and we obtain directly the same result as that obtained from the above formula

$$F(\rho_1, \rho_2) = \frac{1}{\sqrt{\det((\mathbf{A}_1 + \mathbf{A}_2)/2)}} \exp[-\mathbf{u}^T(\mathbf{A}_1 + \mathbf{A}_2)^{-1}\mathbf{u}]. \tag{18}$$

The result of Twamley is re-obtained for $\mathbf{u} = 0$ in a more compact form which is independent of the parametrization. We note that due to (F4) formula (17) is valid for any two displaced squeezed thermal states. In order to compare our result with that of Twamley [8] we shall use the canonical decomposition of any correlation matrix (i.e. of any positive definite matrix) \mathbf{A} (obtained with the aid of a theorem of Balian, De Dominicis and Itzykson [17] concerning the canonical decomposition of symplectic matrices) (see also [11])

$$\mathbf{A} = \mathbf{O}^T\mathbf{M}\mathbf{\Gamma}\mathbf{M}\mathbf{O} \tag{19}$$

where $\mathbf{O} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$, with $c = \cos \theta$ and $s = \sin \theta$, $\mathbf{M} = \begin{pmatrix} m & 0 \\ 0 & 1/m \end{pmatrix}$ and $\mathbf{\Gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$. Then $\det \mathbf{A} = \gamma^2$ and

$$\Delta + T = \gamma_1^2\gamma_2^2 + 1 + \gamma_1\gamma_2 \left[S^2 \left((m_1m_2)^2 + \frac{1}{(m_1m_2)^2} \right) + C^2 \left(\left(\frac{m_1}{m_2} \right)^2 + \left(\frac{m_2}{m_1} \right)^2 \right) \right] \tag{20}$$

where $C = \cos(\theta_2 - \theta_1)$ and $S = \sin(\theta_2 - \theta_1)$. The correspondence between the parametrization from [8] and our parametrization is given by $\cosh r = \frac{1}{2}(m + (1/m))$ and $\cosh(\beta/4) = \gamma/(\gamma^2 - 1)^{1/2}$. Now we consider the exponential factor $\mathcal{F} = \exp[-\mathbf{u}^T(\mathbf{A}_1 + \mathbf{A}_2)^{-1}\mathbf{u}]$. The first remark concerns the following form of $(\mathbf{A}_1 + \mathbf{A}_2)^{-1}$

$$(\mathbf{A}_1 + \mathbf{A}_2)^{-1} = \mathbf{O}_1^T\mathbf{M}_1^{-1}(\mathbf{\Gamma}_1 + \tilde{\mathbf{\Gamma}}_2)^{-1}\mathbf{M}_1^{-1}\mathbf{O}_1 \tag{21}$$

where $\tilde{\mathbf{\Gamma}}_2 = \mathbf{M}_1^{-1}\mathbf{O}_1\mathbf{O}_2^T\mathbf{M}_2\mathbf{\Gamma}_2\mathbf{M}_2\mathbf{O}_2\mathbf{O}_1^T\mathbf{M}_1^{-1}$. Then

$$\mathcal{F} = \exp[-\tilde{\mathbf{u}}^T(\mathbf{\Gamma}_1 + \tilde{\mathbf{\Gamma}}_2)^{-1}\tilde{\mathbf{u}}] \tag{22}$$

where $\tilde{u} = \mathbf{M}_1^{-1} \mathbf{O}_1 u$. The matrix elements of $\mathcal{G} = (\mathbf{\Gamma}_1 + \tilde{\mathbf{\Gamma}}_2)^{-1}$ are given by

$$\begin{aligned} \mathcal{G}_{\tilde{\alpha}\tilde{\alpha}} &= \frac{\gamma_1 + \gamma_2(S^2(m_1 m_2)^2 + C^2(m_1/m_2)^2)}{\Delta} \\ \mathcal{G}_{\tilde{\tau}\tilde{\tau}} &= \frac{\gamma_1 + \gamma_2((S^2/(m_1 m_2)^2) + C^2(m_2/m_1)^2)}{\Delta} \\ \mathcal{G}_{\tilde{\alpha}\tilde{\tau}} &= -\frac{\gamma_2 C S(m_2^2 - (1/m_2^2))}{\Delta}. \end{aligned} \quad (23)$$

When $\theta_1 = \theta_2$ and $m_1 = m_2 = 1$ we re-obtain the result from [9].

4. Conclusions

In this paper we have developed a completely new method for the treatment of the complicated square-root factors from the Uhlmann formula for the transition probability, in the case of Gaussian density operators. This method can be applied to multimode states. Also it can be very useful for the computation of various mean values of the exponential operators whose exponents are at most quadratic in coordinate and momentum operators [15].

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